Automorphism Groups of Partially Ordered Sets

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Pomona Research in Mathematical Experience

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Preliminaries

Poset

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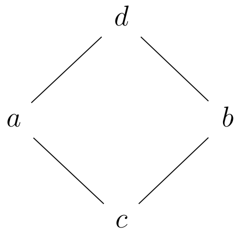
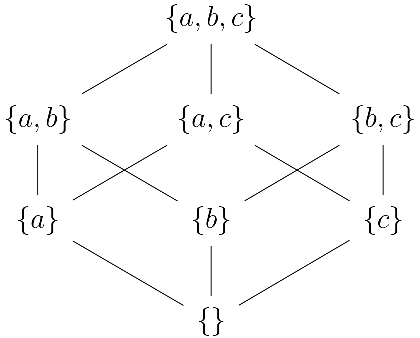
Definition

A ***poset*** $(P, \leq_P)$ *is a set* P *with a relation* $\leq_P$ *such that for all* $x, y, z \in P$*, we have* $x \leq_P x$; $x \leq_P y$ *and* $y \leq_P x \implies y = x$; *and* $x \leq_P y$ *and* $y \leq_P z \implies x \leq_P z$.





Poset examples

Poset Map

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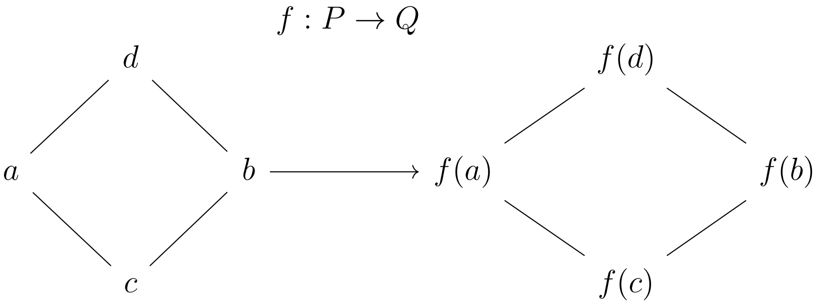
Definition

If P and Q are posets, then $f : P \to Q$ is a ***poset map*** *if for all $x, y \in P$, $x \leq_P y \implies f(x) \leq_Q f(y)$.*



Poset Isomorphism

Definition

A poset map $f : P \to Q$ is said to be ***order-reflexive*** *if for all $x, y \in P$, $f(x) \leq_Q f(y) \implies x \leq_P y$.*

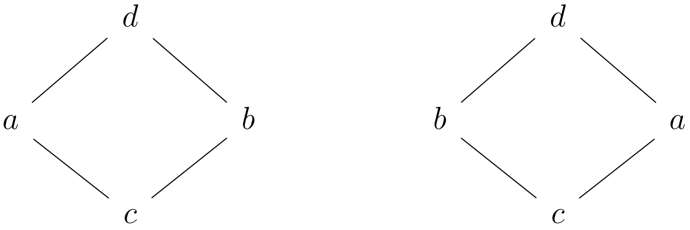
Definition

A surjective, order-reflexive map $f : P \to Q$ is called a ***poset isomorphism***.

Automorphism

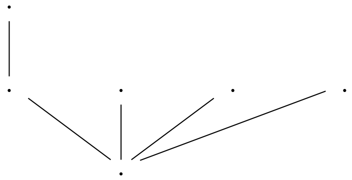
Definition

An isomorphism $f : P \to P$ is called an ***automorphism****. The set of all automorphisms is denoted as $\operatorname{Aut} P$ and is a group under composition. For any poset P with p points, $\operatorname{Aut} P \leq p!$*

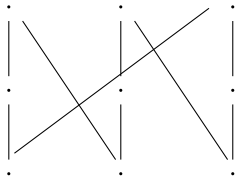


$$\operatorname{Aut} P = Z_2$$

Automorphism Examples



$\mathrm{Aut}\, Q = S_3$



$\mathrm{Aut}\, R = Z_3$
(Barmak '09)

Barmak and Minian 2009

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Theorem (Barmak, Minian '09)

If G is a finite group of order n with r elements, then there exists a poset P with $n(r+2)$ points such that $\mathrm{Aut}\, P \cong G$.

Theorem (Barmak '20)

If G is a finite group, then there exists a poset P of $4|G|$ points such that $\mathrm{Aut}\, P \cong G$.

History

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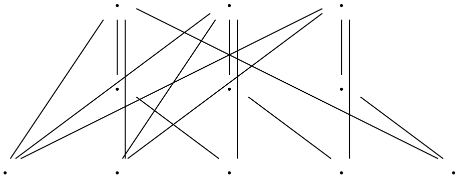
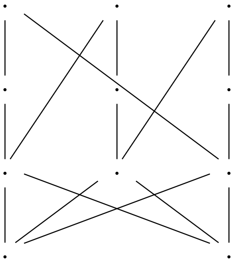
- 1939 - Robert Frucht proves that if G is a finite group of order n with r generators, then there exists a poset P with $n(r+2)$ points such that $\operatorname{Aut} P \cong G$. His paper was translated into English in 1949.
- 1946 - George Birkhoff proves that any finite group G is the automorphism group of some poset P.
- 1972 - M.C. Thornton proves that if G is a finite group of order n with r generators, then there exists a poset P with $n(2r+1)$ points such that $\operatorname{Aut} P \cong G$.
- 2009 - Barmak and Minian unknowingly reprove Frucht's theorem
- 2020 - Barmak proves that the poset P has $4|G|$ points





Beta Values of Groups

Definition

$\boldsymbol{\beta(G)} = \min\{|P| : P$ *is a poset with* $\operatorname{Aut} P \cong G\}$.

Beta Values of Groups

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- $p = 2$ $\quad 2^{k+1} \leq \beta(Z_{2^k}) \leq 2^{k+1} + 12$ if $k \geq 2$
- $p = 3, 5$ $\quad 2p^k \leq \beta(Z_{p^k}) \leq 2p^k + 3p$
- $p \geq 7$ $\quad 2p^k \leq \beta(Z_{p^k}) \leq 2p^k + p$

Realizing Finite Groups G with a Minimal Poset P

$\beta(Z_3)$

Theorem (Barmak, '20)

$\beta(Z_3) = 9$

Definition

*An **orbit** of an element x is all possible destinations of x under group action.*

$\beta(Z_3)$

Theorem (Barmak, '20)

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Definition

*An **orbit** of an element x is all possible destinations of x under group action.*

2 orbits of 3 $\implies$ $\mathrm{Aut}\, P \neq Z_3$

3 orbits of 3 $\implies$ $\mathrm{Aut}\, P = Z_3$

$\beta(S_n)$

Proposition (C., G., M., O., S., '22)

$\beta(S_n) = n$ for all $n \in \mathbb{N}$

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$\beta(S_n)$

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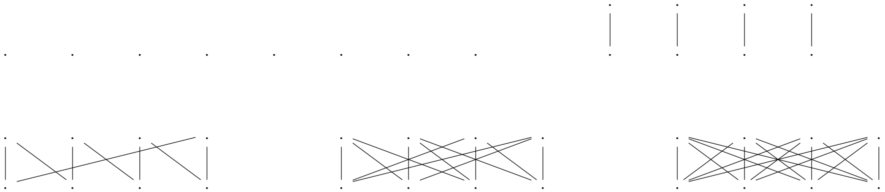
- $\mathrm{Aut}\, P \cong S_n$
- $\beta(S_n) = n$ for all $n \in \mathbb{N}$

Conjectures

Conjecture

There is a nice proof of $\beta(Z_4) = 12$



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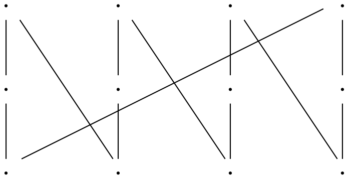


Conjectures

Conjecture

$\beta(Z_p) = 3p$ for any prime p

Conjecture

$\beta(Z_{p^k}) = 2p^k + p$ for all primes $p \geq 7$

Realizing Finitely Generated Abelian Groups

Realizing Infinite Groups

Previous work has realized all finite groups (Barmak '09)

Motivating Question

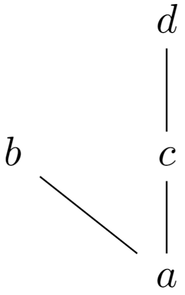
Can we realize finitely generated abelian groups?

The simplest relevant example is $\mathbb{Z}$

Covers

Definition

Let $x, y \in P$. We say that y covers x if $x \leq_P y$ and there is no $z \in P$ such that $x <_P z <_P y$.



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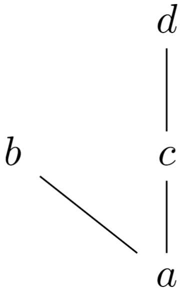
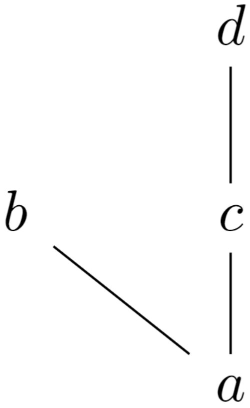


b and c cover a, but d does not cover a

Covers

Covers are preserved under automorphisms



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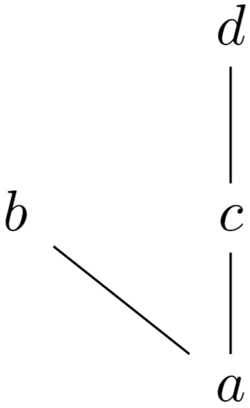

Proposition

Let $f \in \mathrm{Aut}\, P$. If b covers a then $f(b)$ covers $f(a)$

Covers

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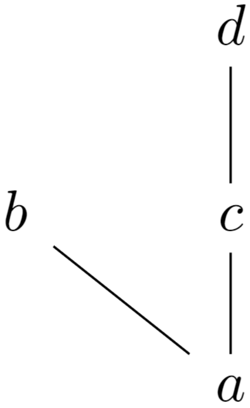

Proposition

Let $f \in \mathrm{Aut}\, P$. If b covers a then $f(b)$ covers $f(a)$

Proof Sketch: Since f is order reflexive, $f(a) < z < f(b)$ implies $a < f^{-1}(z) < b$

Realizing the Integers

$$\begin{array}{c} \vdots \\ \uparrow \\ 1 \\ \uparrow \\ 0 \\ \uparrow \\ -1 \\ \uparrow \\ \vdots \end{array}$$

Realizing the Integers

$$\vdots$$
$$\uparrow$$
$$1$$
$$\uparrow$$
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$$\uparrow$$
$$-1$$
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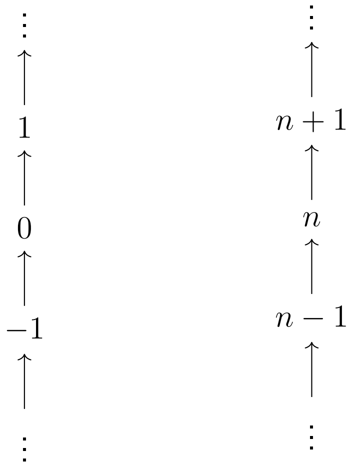
Result

Let $P = (\mathbb{Z}, \leq)$ with the usual order $\leq$. Then $\operatorname{Aut} P \cong \mathbb{Z}$

Realizing the Integers



Result

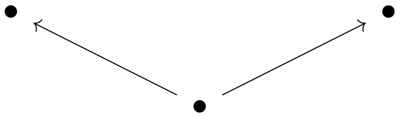
Let $P = (\mathbb{Z}, \leq)$ with the usual order $\leq$. Then $\operatorname{Aut} P \cong \mathbb{Z}$

The only automorphisms of P are $f(x) = x + n$ for some $n \in \mathbb{Z}$

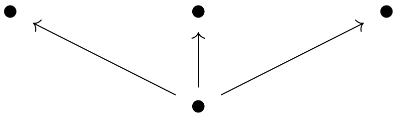
Going Further

Motivating Question

Having realized G and H, how do we realize $G \times H$?



Aut $P \cong \mathbb{Z}_2$



Aut $Q \cong S_3$

Examples

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What is $\mathrm{Aut}\, P \sqcup Q$?

Examples

Aut $P \cong \mathbb{Z}_2$

Aut $Q \cong \mathbb{Z}_2$

$$\mathrm{Aut}\, P \sqcup Q \cong S_4 \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$$

Connected Posets

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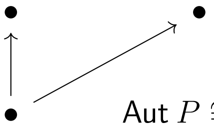
Definition

A poset P is connected if for any $x, y \in P$, there exists a sequence of points $(x = \gamma_1, \gamma_2, ..., \gamma_n = y)$ in P such that γ_i is comparable to γ_{i+1} for all $1 \leq i < n$.

Examples



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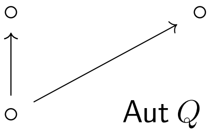

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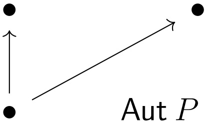

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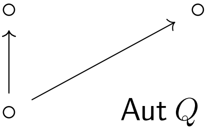
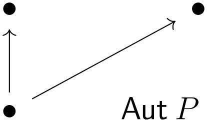


Aut $Q \cong \mathbb{Z}_2$

What is $\mathrm{Aut}\, P \sqcup Q$?

Examples



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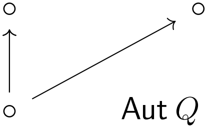


Aut $Q \cong \mathbb{Z}_2$

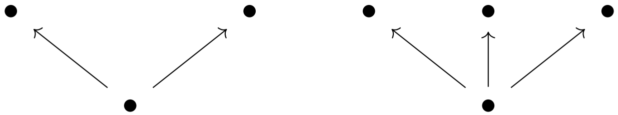
$$\text{Aut } P \sqcup Q \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$$

A Useful Result

Theorem (C., G., M., O., S., '22)

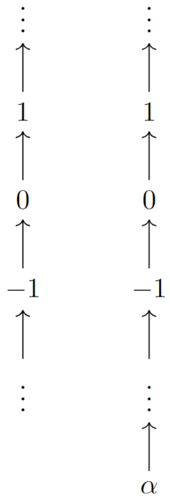
Let $\mathcal{P} = \sqcup P_i$ be a disjoint union of nonempty posets, where each P_i is connected, and they are pairwise not isomorphic. Then,

$$\operatorname{Aut}\mathcal{P} \cong \prod \operatorname{Aut} P_i$$



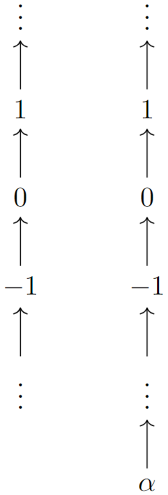
$$\text{Aut} P \sqcup Q \cong \mathbb{Z}_2 \times S_3$$

Realizing $\mathbb{Z}^r$



Let $P_0 = (\mathbb{Z}, \leq)$ and $P_1 = P_0 \cup \{\alpha\}$.

Realizing $\mathbb{Z}^r$



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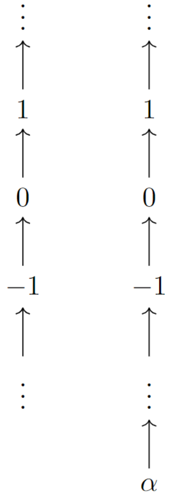

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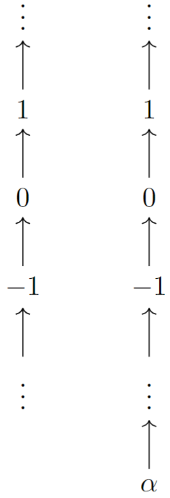

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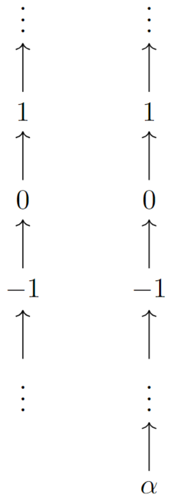

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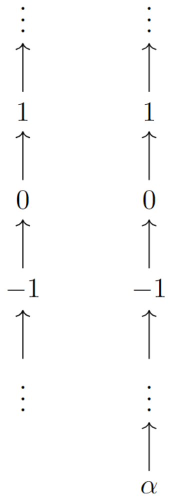
$$\operatorname{Aut} \bigsqcup_{0<i<r} P_i \cong \mathbb{Z}^r$$

Realizing $\mathbb{Z}^r$



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Result

$\mathbb{Z}^r$ is realizable for all $r \in \mathbb{N}$.

Realizing All F.G.A.G.

Theorem

Every finitely generated abelian group G can be decomposed as

$$G \cong \mathbb{Z}^r \times \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \ldots \times \mathbb{Z}_{n_k}$$

Main Result

- We know all F.G.A.G are isomorphic to $\mathbb{Z}^r \times T$.

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- $\mathbb{Z}^r$ is realizable

Theorem (C., G., M., O., S., '22)

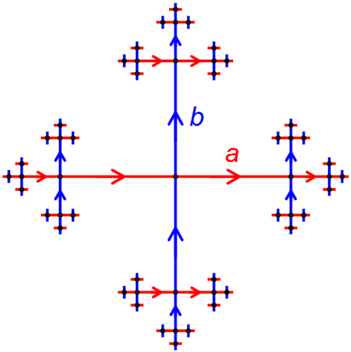
Every finitely generated abelian group is realizable as the automorphism group of some poset

Free Groups

Definition

The free group F_S over the set S is the set of reduced words that can be built from elements of S

- Let $S = \{a, b\}$. Then $ab^2a \in S$
- We have inverses: $a^{-1}, b^{-1} \in S$
- $bbaa^{-1} = b^2$

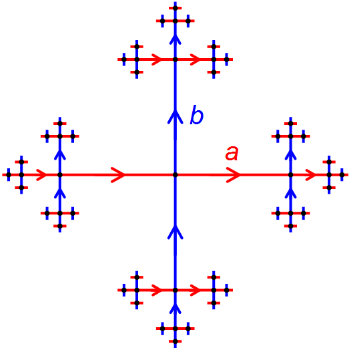


A Partial Order On Free Groups

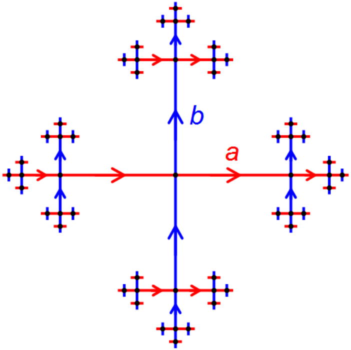
Theorem (C., G., M., O., S., '22)

Let $x, y \in F_S$. The relation $x \leq y \iff y = xg_1g_2...g_k$ where $g_i \in S \cup \{1_{F_S}\}$ makes F_S into a poset.

- $a \leq a$ since $1_{F_S} \in S \cup \{1_{F_S}\}$
- $a \leq ab^2$ since $b \in S \cup \{1_{F_S}\}$
- $a^{-7} \leq a^{-7}a^7 = 1_{F_S}$
- b is not comparable to a



Automorphism Group Of Free Group



Consider the map $f_p(x) = px$ for some $p \in F_S$. We proved that this is a poset automorphism. Take f_a for example:

- $b^{-1} \leq a \iff ab^{-1} \leq a^2$

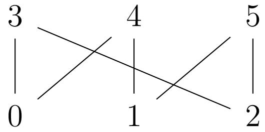
Result

There exists a poset P such that $\operatorname{Aut} P$ contains an isomorphic copy of F_S for any free group F_S

Counting Automorphisms with Python

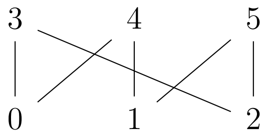
Counting Automorphisms

How many automorphisms does this poset have?



Counting Automorphisms

How many automorphisms does this poset have?



We can use Python to find out

Counting Automorphisms

How does the code work?

Counting Automorphisms

How does the code work?

1. Finds S_P for a poset P

Counting Automorphisms

How does the code work?

1. Finds S_P for a poset P
2. Finds and counts all of the bijections that preserve the structure of the poset

How Do We Count Automorphisms?

Why do we find S_P?

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Lemma

If P is a poset, then Aut P is a subgroup of S_P.

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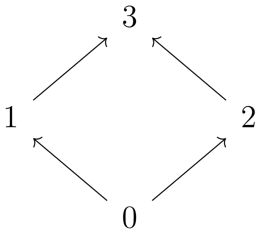

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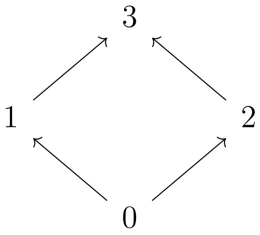
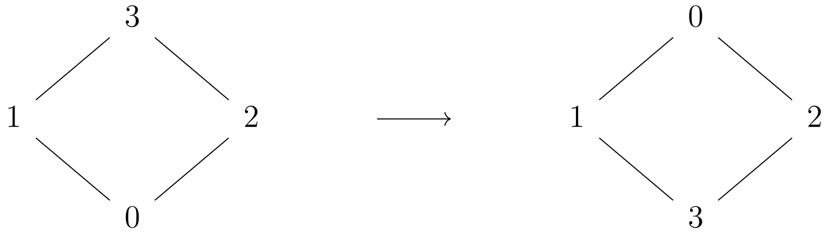
Lemma

If P is a poset, then Aut P is a subgroup of S_P.



Aut$P$ is a subgroup of S_4

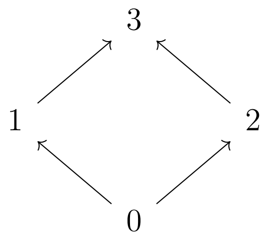
How Do We Count Automorphisms?



Getting Bijections from S_P

Let $P := \{0, 1, 2, 3\}$.
We focus on S_4.
e.g. $(0\ 3) \in S_P$
This is the same as the mapping $0 \mapsto 3$ and $3 \mapsto 0$.
This is not an automorphism

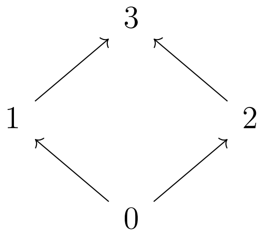
How Do We Count Automorphisms?



Checking to See Which Bijections are Automorphisms

We use matrices.
We construct $C = [c_{ij}]$ where $i, j \in P$. We set $c_{ij} = 1$ if $i = j$ or if j covers i. Otherwise, $c_{ij} = 0$.

How Do We Count Automorphisms?



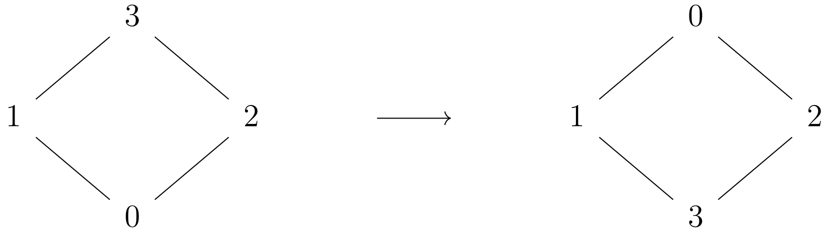
$$
\begin{array}{cccc}
0 & 1 & 2 & 3
\end{array}
$$

$$
\begin{pmatrix}
1 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{array}{c}
0 \\ 1 \\ 2 \\ 3
\end{array}
$$

How Do We Count Automorphisms?

Applying a Bijection to the Matrix

Based on the bijection mappings, we swap the rows and columns of the matrix.
Consider the bijection from earlier
$0 \mapsto 3$ and $3 \mapsto 0$



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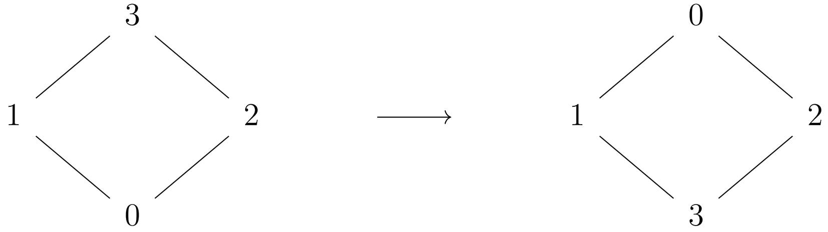

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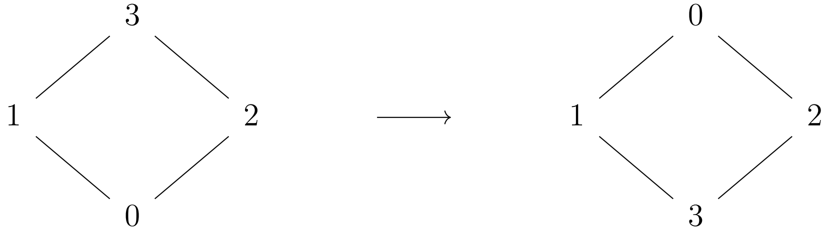

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \rightarrow \begin{array}{c} \\ 3 \\ 1 \\ 2 \\ 0 \end{array}\!\!\begin{array}{c} \begin{matrix} 3 & 1 & 2 & 0 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

How Do We Count Automorphisms?

When is a Bijection an Automorphism?

A bijection is an automorphism when the bijection does not change the matrix.

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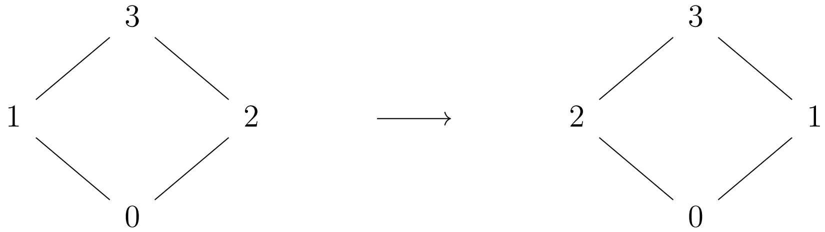
Example:



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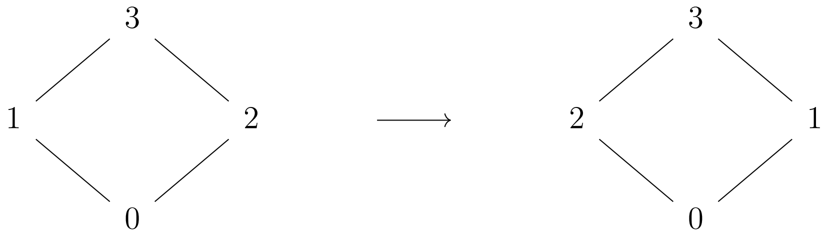

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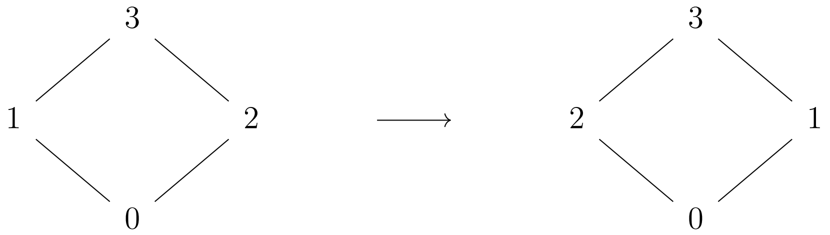

$$
\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}
\begin{array}{c} \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}
\to
\begin{array}{c} \\ 0 \\ 2 \\ 1 \\ 3 \end{array}
\begin{array}{c} \begin{matrix} 0 & 2 & 1 & 3 \end{matrix} \\ \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}
$$

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Putting it All Together

The program finds $\mathrm{Aut}\, P$ by

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1. Finding S_P

How Do We Count Automorphisms?

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Putting it All Together

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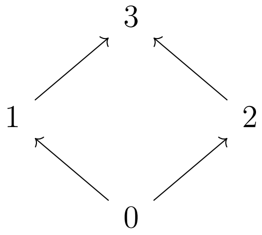
Running the Code



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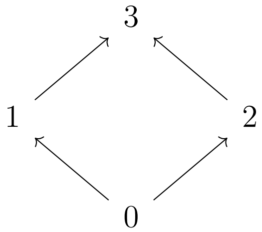
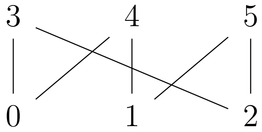



```
This poset has 2 automorphisms

The automorphisms on this poset are :

f0:

0 -> 0
1 -> 1
2 -> 2
3 -> 3


f1:

0 -> 0
1 -> 2
2 -> 1
3 -> 3
```

Running the Code



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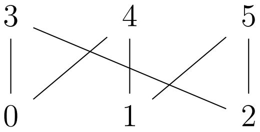



```
This poset has 6 automorphisms

The automorphisms on this poset are :

f0:              f3:

0 -> 0           0 -> 1
1 -> 1           1 -> 2
2 -> 2           2 -> 0
3 -> 3           3 -> 4
4 -> 4           4 -> 5
5 -> 5           5 -> 3


f1:              f4:

0 -> 0           0 -> 2
1 -> 2           1 -> 0
2 -> 1           2 -> 1
3 -> 4           3 -> 5
4 -> 3           4 -> 3
5 -> 5           5 -> 4


f2:              f5:

0 -> 1           0 -> 2
1 -> 0           1 -> 1
2 -> 2           2 -> 0
3 -> 5           3 -> 3
4 -> 4           4 -> 5
5 -> 3           5 -> 4
```

Running the Code

For posets P of the following form,

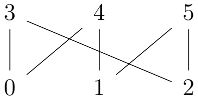


Figure: n = 3

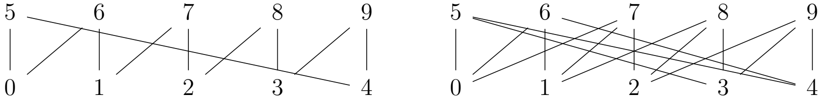


Figure: n = 5

Running the Code

For posets P of the following form,

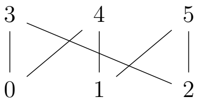


Figure: n = 3

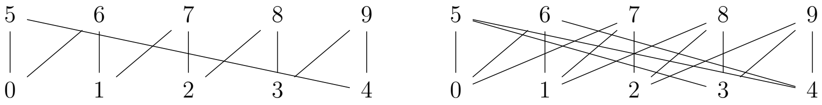
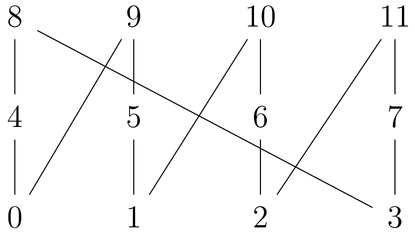


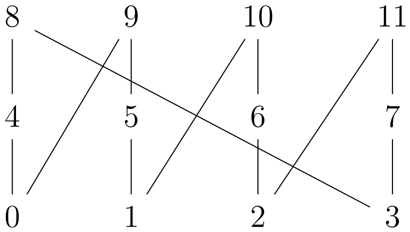
Figure: n = 5

we think that Aut $P = D_{2n}$ for $3 \leq n \leq 7$

Running the Code

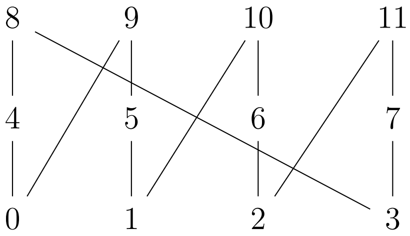


Running the Code



This takes too long and does not provide an output.
Why does this happen and how to we fix it?

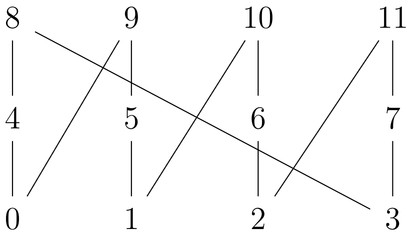
Making the Code More Efficient



$|S_{12}| = 12! = 479001600.$

The program has to check a lot of automorphisms.

Making the Code More Efficient



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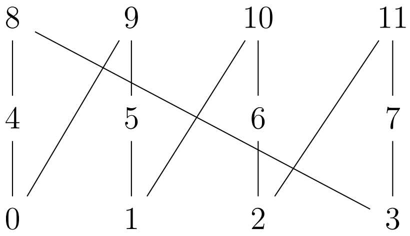

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We can add restrictions.

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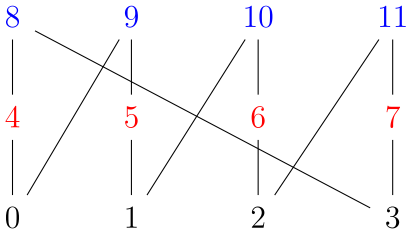

$|S_{12}| = 12! = 479001600.$
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We find all of the bijections for each height and their combinations

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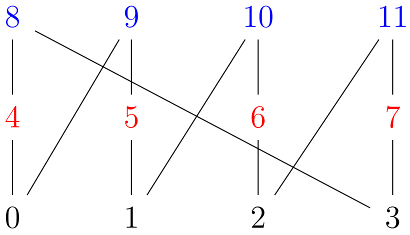
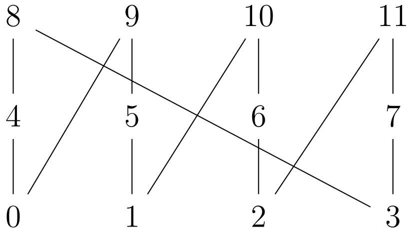


$|S_{12}| = 12! = 479001600.$
The program has to check a lot of automorphisms.

How do we reduce the number of things the program must check?
We can add restrictions.

We find all of the bijections for each height and their combinations
We only need to check $|S_4| \times |S_4| \times |S_4| = 13824$ automorphisms.

Running the Code



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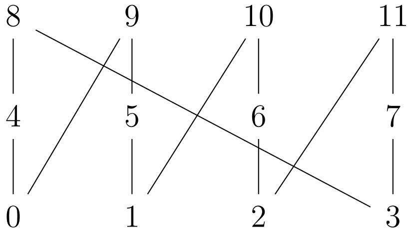


```
This poset has 4 automorphisms
The automorphisms on this poset are :

f0:

0 -> 0
1 -> 1
2 -> 2
3 -> 3
4 -> 4
5 -> 5
6 -> 6
7 -> 7
8 -> 8
9 -> 9
10 -> 10
11 -> 11

f1:

0 -> 1
1 -> 2
2 -> 3
3 -> 0
4 -> 5
5 -> 6
6 -> 7
7 -> 4
8 -> 9
9 -> 10
10 -> 11
11 -> 8

f2:

0 -> 2
1 -> 3
2 -> 0
3 -> 1
4 -> 6
5 -> 7
6 -> 4
7 -> 5
8 -> 10
9 -> 11
10 -> 8
11 -> 9

f3:

0 -> 3
1 -> 0
2 -> 1
3 -> 2
4 -> 7
5 -> 4
6 -> 5
7 -> 6
8 -> 11
9 -> 8
10 -> 9
11 -> 10
```

Future Work

- Conjectures of $\beta(Z_{p^k})$
- The free group of rank r is realizable for all $r > 1$
- Construction of posets P such that $\mathrm{Aut}\, P = D_{2P}$
- Extending the code to identity $\mathrm{Aut}\, P$ for a poset P

Acknowledgements

We would like to acknowledge and thank the NSF[1], Cory Colbert, Rodrigo Smith, Edray Goins, Alex Barrios, and all other professors and students who have been part of PRiME '22. We would also like to thank the NSF and Pomona College.

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References

Jonathan A. Barmak.
Automorphism groups of finite posets ii.
arXiv, 2020.

Jonathan Ariel Barmak and Elias Gabriel Minian.
Automorphism groups of finite posets.
Discrete Mathematics, 309(10):3424–3426, 2009.

G. Birkoff.
On groups of automorphisms.
Rev. Un. Math. Argentina, 11:155–157, 1946.

Robert Frucht.
Herstellung von graphen mit vorgegebener abstrakter gruppe.
Compositio Mathematica (German), 6:239–250, 1939.

M.C. Thorton.
Spaces with given homeomorphism groups.
Proceedings of the American Mathematical Society, 33:127–131, 1972.

Thank You!